POINTED HOPF ALGEBRAS OVER SOME SPORADIC SIMPLE GROUPS

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Abstract. Any finite-dimensional complex pointed Hopf algebra with group of group-likes isomorphic to a sporadic group, with the possible exception of the Fischer group Fi_{22}, the Baby Monster B and the Monster M, is a group algebra.

1. Introduction

Let $\Bbbk$ be an algebraically closed field of characteristic 0. In this Note, we announce a new contribution to the classification of finite-dimensional Hopf algebras over $\Bbbk$. As is known, different classes of finite-dimensional Hopf algebras have to be studied separately because the pertaining methods are radically different. There is a method for pointed Hopf algebras (those whose coradical is a group algebra $\Bbbk G$) that has been applied with satisfactory results when G is abelian [8]; an exposition of the method can be found in [7]. Recently, it appeared that many finite simple (or almost simple) groups G admit very few finite-dimensional, pointed Hopf algebras with coradical isomorphic to $\Bbbk G$:

- Any finite-dimensional complex pointed Hopf algebra with group of group-likes isomorphic to $\mathbb{A}_m$, $m \geq 5$, is a group algebra [2].
- Same for the groups $SL(2, 2^n)$, $n > 1$ [10] and M_{20}, $M_{21} = PSL(3, 4)$ [11].
- Most of the pointed Hopf algebras over the symmetric groups have infinite dimension, with the exception of a short list of open possibilities, see [2, 4] and references therein. More precisely, most of the irreducible Yetter-Drinfeld modules have infinite-dimensional Nichols algebras (see below).

This is a report on finite-dimensional pointed Hopf algebras over sporadic simple groups. As part of our results, we have the following.

Theorem 1. *Let G be any sporadic simple group, different from the Fischer group Fi_{22}, the Baby Monster B and the Monster M. If H is a finite-dimensional pointed Hopf algebra with $G(H) \simeq G$, then $H \simeq \Bbbk G$.*

The Theorem holds more generally over any field of characteristic 0, since the property of being pointed is stable under extension of scalars.

1.1. **Glossary.** For the reader's convenience, we recall a few definitions that are central to our work. More information can be found in [5, 7]. Let H be a Hopf algebra with comultiplication Δ and bijective antipode $\mathcal{S}$.

• An element $g \neq 0$ in H is a *grouplike* if $\Delta(g) = g \otimes g$; the set of all grouplikes is a group $G(H)$ with multiplication given by the product of H.

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• A *Yetter-Drinfeld module* over H is a left H-module M that bears also a structure $\lambda : M \to H \otimes M$ of H-comodule, compatible with the action in an appropriate sense. If H is finite-dimensional, then a Yetter-Drinfeld module is the same as a module over the Drinfeld double of H. For instance, if $H = \Bbbk G$ is the group algebra of a finite group G, then a Yetter-Drinfeld module over H is a left G-module M that bears also a G-gradation $M = \oplus_{g\in G} M_g$, compatibility meaning that $h \cdot M_g = M_{hgh^{-1}}$ for all $h, g \in G$.

• A *rack* is a pair $(X, \triangleright)$ where X is a non-empty set and $\triangleright : X \times X \to X$ is an operation such that the map $\varphi_x = x \triangleright \underline{\quad}$ is bijective for any $x \in X$, and $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ for all $x, y, z \in X$. A map $q : X \times X \to GL(n, \Bbbk)$ is a *2-cocycle* of degree n if

$$q_{x,y\triangleright z} q_{y,z} = q_{x\triangleright y, x\triangleright z} q_{x,z}, \quad \text{for all } x, y, z \in X.$$

• A *braided vector space* is a pair (V, c) where V is a vector space and $c \in GL(V \otimes V)$ fulfills the braid equation: $(c\otimes \mathrm{id})(\mathrm{id} \otimes c)(c\otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c\otimes \mathrm{id})(\mathrm{id} \otimes c)$. Examples:

(i) Any Yetter-Drinfeld module is a braided vector space in a natural way.

(ii). Let X be a finite rack, q a 2-cocycle of degree n, $V = \Bbbk X \otimes \Bbbk^n$, where $\Bbbk X$ is the vector space with basis e_x, $x \in X$. We denote $e_x v := e_x \otimes v$. Consider the linear isomorphism $c^q : V \otimes V \to V \otimes V$, $c^q(e_x v \otimes e_y w) = e_{x\triangleright y} q_{x,y}(w) \otimes e_x v$, $x, y \in X$, $v, w \in \Bbbk^n$. Then (V, c^q) is a braided vector space.

The braided vector spaces arising as Yetter-Drinfeld modules over group algebras of finite groups can be presented in terms of racks and cocycles, see a bit more of information below.

• We assume the reader familiar with the important notion of the *Nichols algebra* of a braided vector space, discussed at length in [7]. In short, one of the possible definitions of the Nichols algebra $\mathfrak{B}(V)$ of a braided vector space (V, c) is as follows. Since c satisfies the braid equation, it induces a representation of the braid group $\mathbb{B}_n$, $\rho_n : \mathbb{B}_n \to GL(V^{\otimes n})$, for each $n \geq 2$. Let $Q_n = \sum_{\sigma\in\mathbb{S}_n} \rho_n(M(\sigma)) \in End(V^{\otimes n})$, where $M : \mathbb{S}_n \to \mathbb{B}_n$ is the so-called Matsumoto section (not a morphism of groups, but preserves product when length is preserved). Then $\mathfrak{B}(V)$ is the quotient of the tensor algebra $T(V)$ by $\oplus_{n\geq 2} \ker Q_n$, in fact a 2-sided ideal of $T(V)$. If c is the usual switch, then $\mathfrak{B}(V)$ is just the symmetric algebra of V; but in general the determination of a Nichols algebra is quite a difficult task.

2. Outline of the proof

A complete proof of Theorem 1 for the groups M_{22} and M_{24} is contained in [9]; the proof for the other groups is included in [3].

We sketch now the proof in two main reductions. The first one has been explained in several places, with detail in [7], but we include a brief summary for completeness. We remind that if U is a braided vector subspace of V, then $\mathfrak{B}(U) \hookrightarrow \mathfrak{B}(V)$.

2.1. **A general reduction.** Let G be a finite group, H a pointed Hopf algebra with $G(H) \simeq G$. Then there are two basic invariants of H, a Yetter-Drinfeld module V over $\Bbbk G$ (called the infinitesimal braiding of H) and its Nichols algebra $\mathfrak{B}(V)$. We have $|G| \dim \mathfrak{B}(V) \leq \dim H$. Therefore, the following statements are equivalent:

(1) If H is a finite-dimensional pointed Hopf algebra with $G(H) \simeq G$, then $H \simeq \Bbbk G$.

(2) If $V \neq 0$ is a Yetter-Drinfeld module over $\Bbbk G$, then $\dim \mathfrak{B}(V) = \infty$.

(3) If V is an *irreducible* Yetter-Drinfeld module over $\Bbbk G$, then $\dim \mathfrak{B}(V) = \infty$.

2.2. **Looking at subracks.** We focus on (3) above. The second reduction has been the basis of our recent papers. It starts from the well-known classification of irreducible Yetter-Drinfeld modules over $\Bbbk G$ by pairs $(\mathcal{O},\rho)$, where $\mathcal{O}$ is a conjugacy class in G and ρ is an irreducible representation of the stabilizer G^s of a fixed point $s\in\mathcal{O}$. Now, the definition of the Nichols algebra $\mathfrak{B}(\mathcal{O},\rho)$ of the corresponding Yetter-Drinfeld module $M(\mathcal{O},\rho)$ just depends on the braiding. If $\dim\rho=1$, then this braiding depends only on the *rack* $\mathcal{O}$ and a 2-cocycle $q:\mathcal{O}\times\mathcal{O}\to\Bbbk^\times$ [5]. Namely, $\mathcal{O}$ is a rack with the product $x\triangleright y:=xyx^{-1}$, $M(\mathcal{O},\rho)$ has a natural basis $(e_x)_{x\in\mathcal{O}}$ and the braiding is given by $c(e_x\otimes e_y)=q_{xy}e_{x\triangleright y}\otimes e_x$. If there exists a subrack X of $\mathcal{O}$ such that the Nichols algebra of the braided vector space defined by X and the restriction of q is infinite dimensional, then $\dim\mathfrak{B}(\mathcal{O},\rho)=\infty$.

We recall some examples of racks which are relevant in this work.

(i) Abelian racks: those racks X such that $x\triangleright y=y$ for all $x,y\in X$.

(ii) $\mathcal{D}_p$: the class of involutions in the dihedral group $\mathbb{D}_p$ (of order $2p$), p a prime.

(iii) $\mathfrak{O}$: the class of 4-cycles in $\mathbb{S}_4$.

(iv) Doubles of racks: if X is a rack, then $X^{(2)}$ denotes the disjoint union of two copies of X each acting on the other by left multiplication.

We are interested in finding subracks which are abelian, or isomorphic to $\mathcal{D}_p^{(2)}$ or to $\mathfrak{O}^{(2)}$, by the following reasons:

(A) If X is abelian, then the corresponding braided vector space is of diagonal type. Braided vector spaces of diagonal type with finite-dimensional Nichols algebra where classified in [13]; thus, we just need to check if the matrix (q_{xy}) belongs or not to the list in [13].

(B) If X is isomorphic either to $\mathcal{D}_p^{(2)}$ or to $\mathfrak{O}^{(2)}$, then for some specific cocycles, the related Nichols algebras have infinite dimension [6, Ths. 4.7, 4.8].

Variations.

(a) If $\dim\rho>1$, similar arguments apply.

(b) Sometimes the rack X is not abelian, but the braided vector space produced by X and the 2-cocycle can be realized with an abelian rack, by a suitable change of basis.

(c) Let $F<G$ be a subgroup, $s\in F$, $\mathcal{O}^F$, resp. $\mathcal{O}^G$ the conjugacy class of s in F, resp. in G. If $\dim\mathfrak{B}(\mathcal{O}^F,\tau)=\infty$ for any irreducible representation τ of F^s, then $\dim\mathfrak{B}(\mathcal{O}^G,\rho)=\infty$ for any irreducible representation ρ of G^s.

(d) A conjugacy class $\mathcal{O}$ is real if $\mathcal{O}=\mathcal{O}^{-1}$. It is quasireal if $\mathcal{O}=\mathcal{O}^m$ for some integer m, $1<m<N$, where N is the order of the elements in $\mathcal{O}$. The search of subracks isomorphic to $\mathcal{D}_p^{(2)}$ or to $\mathfrak{O}^{(2)}$, as well as the verification that the restriction of the cocycle q is as needed in (B), is greatly simplified in a real (quasireal) conjugacy class [1].

(e) We say that a rack X is of type D if there exists a decomposable subrack $Y=R\coprod S$ of X such that $r\triangleright(s\triangleright(r\triangleright s))\neq s$, for some $r\in R$, $s\in S$. If a conjugacy class $\mathcal{O}$ is a rack of type D, then $\dim\mathfrak{B}(\mathcal{O},\rho)=\infty$ for any ρ (see [2] and Theorem 8.6 of [14]).

2.3. **Computations.** We now fix a sporadic group G as in Theorem 1. We extracted relevant information from the ATLAS [15] with the `AtlasRep` package [16]. Then, we checked when a conjugacy class is real or quasireal or of type D. We used GAP [12] for the computations.

These tools allow to apply the techniques sketched above to all pairs $(\mathcal{O},\rho)$ and establish the validity of (3).

2.4. Some of these results were announced in several meetings:

- Hopf Algebras and Related Topics, A conference in honor of Professor Susan Montgomery. University of Southern California, Los Angeles, USA. February 2009.
- IV Encuentro Nacional de Álgebra, Córdoba, Argentina. August, 2008.
- First De Brún Workshop on Computational Algebra, National University of Ireland, Galway, Ireland. August, 2008
- Groupes quantiques dynamiques et cagories de fusion. CIRM, Luminy, France. April 2008.

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